END SEMESTRAL EXAMINATION
TOPOLOGY
MMATH FIRST YEAR 2021-2022

Time 3 hours

Max. Score 60

Answer all questions.

(1) Let (X, τ) be a topological space, with τ induced by a metric d.
(a) Show that $d : X \times X \longrightarrow \mathbb{R}$ is continuous.
(b) Show that if τ' is another topology on X, such that $d : X \times X \longrightarrow \mathbb{R}$ is continuous with respect to τ', then τ' is finer than τ. (4+6)

(2) Let $K = \{1/n,\ n = 1, 2, \ldots\} \subset \mathbb{R}$. Let $\mathcal{B}$ be the collection of all open intervals (a, b), $a, b \in \mathbb{R}$ along with all sets of the form $(a, b) - K$. Let $\mathbb{R}_K$ denote the set of real numbers with the topology generated by the collection $\mathcal{B}$.
(a) Is $[0, 1]$ a compact subspace of $\mathbb{R}_K$?
(b) Is $\mathbb{R}_K$ path-connected?
Let Y be the quotient space obtained from $\mathbb{R}_K$ by identifying the set K to a point. Let $p : \mathbb{R}_K \to Y$ be the quotient map.
(c) Does Y satisfy the T_1 axiom?
(d) Is Y Hausdorff?
(e) Show that the map $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map. (4×5)

(3) (a) Show that the space $X = [0, 1]^\omega$ with the uniform topology is not limit point compact.
(b) Let (X, d) be a metric space. Let $f : X \to X$ be a function satisfying the condition

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$. Show that if X is compact, then f is a homeomorphism. (5+10)

(4) Let X be a non-compact locally compact Hausdorff space, and let Y be a Hausdorff space. Let $\tilde{X}$ denote the one-point compactification of X and let $\phi : X \to Y$ be a continuous map. Show that the following are equivalent:
(i) There exists a continuous map $\tilde{\phi} : \tilde{X} \to Y$ with $\tilde{\phi}|_X = \phi$.
(ii) There exits some point $y \in Y$, such that for every neighbourhood V of p, there exists some compact subset $C_V \subset X$ such that $\phi(X - C_V) \subset V$. (15)

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